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Neumann Problem of One–Dimensional Nonlinear Thermoelastic Equations

筑波大・数学系
Yoshihiro Shibata 柴田 良弘
Inst. of Math. Univ. of Tsukuba, Tsukuba-shi, 305 Japan

I will report here a joint work [3] with S. Kawashima, Kyushu Univ. about a global in time solvability and an asymptotic stability of Neumann problem of one-dimensional nonlinear thermoelastic equations. This problem was first solved by Shibata [5] in the bounded interval case, but the assumption on the regularity of initial data was not optimal and polynomial decay rate was only shown. And then, Jiang Song [2] improved the result concerning the solvability, regarding an optimal assumption of the regularity of initial data. He also proved an existence theorem globally in time in the half line case and he obtained an exponential decay result in the linear case. In the both works, it is essential that no external force appears. In this note, I essentially assume that the external force depends on the spatial variable only. When the external force depends on not only the spacial variable but also the time, to solve the problem seems to be difficult, in particular such difficulty appears when we decide an asymptotic behaviour of solutions globally in time. Such difficulty sometimes appears in the Neumann case in general and as a technical difficulty, we can not use a Poincaré's inequality.

First, let me formulate the problem. Let $\Omega = (0,1)$ be the unit interval of $\mathbf{R}$ identified with reference body with natural temperature $\tau_0 > 0$. The deformation of the reference body Ω after time t past is described by the deformation map:

$$X : x \in \Omega \longmapsto X(t,x) \in \mathbf{R}.$$

Let $T(t,x)$ be the absolute temperature of the point $X(t,x)$. The equation of motion and the balance of energy are described by the following equations:

$$\rho(x)X_{tt} - S_x = f \qquad \text{for } x \in \Omega \text{ and } t > 0, \tag{1.1}$$

$$\begin{aligned}(e + \frac{\rho(x)}{2}X_t^2)_t - (SX_t)_x \\ = q_x + X_t f + g \qquad \text{for } x \in \Omega \text{ and } t > 0.\end{aligned} \tag{1.2}$$

Here, the subscripts stand for partial derivatives, S is the stress function, e is the internal energy function, q is the heat flux, $\rho(x)$ is the mass density, f is an external force and g is an external heat supply. For simplicity, I assume that

$$\rho(x) = 1, \;\; g = 0 \text{ and } q = \kappa T_x,$$

κ being a positive constant. In this lecture note, as a boundary condition, I consider the traction free and thermally insulated condition which is described by the following formula:

$$S = q = 0 \qquad \text{for } x \in \partial\Omega. \tag{N.N}$$

Note that $\partial\Omega$ consists of only two points 0 and 1. The initial condition is given by the formula:

$$X(0,x) = X_0(x),\ \ X_t(0,x) = X_1(x),\ \ T(0,x) = T_0(x). \tag{1.3}$$

Now, let me explain assumptions. The first one is that

$$f = f(x),\ \ \text{that is } f \text{ depends only on } x. \tag{A.1}$$

Next one is concerned with the constitutive relation. Let F be a variable corresponding to X_x. Let ψ be a Helmholtz free energy function and η be an entropy function. The second assumption is that

(A.2) S, e, ψ and η are functions in (F, T) only, that is

$$S = S(F,T),\ \ e = e(F,T),\ \ \psi = \psi(F,T),\ \ \eta = \eta(F,T),$$

and they are in $C^\infty(G(\delta))$,

where

$$G(\delta) = \{(F,T) \in \mathbf{R}^2 \mid |(F,T) - (1,\tau_0)| \leq \delta\}.$$

In this lecture, all the functions are real–valued unless it is mentioned. The 2nd Law of Thermodynamics tells us that the following two relations are equivalent:

$$de = SdF + Td\eta \iff d\psi = SdF - \eta dT,$$

from which the following constitutive relations follow:

$$S = \frac{\partial\psi}{\partial F},\ \ \eta = -\frac{\partial\psi}{\partial T},\ \ e = \psi - T\frac{\partial\psi}{\partial T}. \tag{A.3}$$

The next assumption is that

$$\frac{\partial^2\psi}{\partial F^2} > 0,\ \ \frac{\partial^2\psi}{\partial F\partial T} \neq 0,\ \ \frac{\partial^2\psi}{\partial T^2} < 0 \ \ \text{in } G(\delta). \tag{A.4}$$

Under the constitutive relation (A.3), (1.2) is equivalent to the following equation:

$$T\eta_t = q_x \qquad \text{for } x \in \Omega \qquad \text{and } t > 0. \tag{1.4}$$

In fact,

$$e_t = \frac{\partial\psi}{\partial T}T_t + \frac{\partial\psi}{\partial F}X_{xt} - T_t\frac{\partial\psi}{\partial T} - T\left(\frac{\partial\psi}{\partial T}\right)_t = T\eta_t + SX_{xt},$$

$$\left(\frac{1}{2}X_t^2\right)_t = X_tX_{tt} = S_xX_t + fX_t.$$

Combining these two equations implies (1.4). Except for finding a conservative quantity, usually we solve (1.1) and (1.4) instead of (1.1) and (1.2).

EXAMPLE 1.1. As a Helmholtz free energy function, let us choose

$$\psi(F,T) = \sqrt{1+F^2} - T^2 - \gamma TF, \ \ \gamma \neq 0.$$

Then,

$$S = F/\sqrt{1+F^2} - \gamma T, \ \ e = \sqrt{1+F^2} + T^2, \ \ \eta = \gamma F + 2T.$$

The corresponding equations are that

$$\begin{cases} X_{tt} - \left(X_x/\sqrt{1+X_x^2} - \gamma T\right)_x = f, \\ T(2T + \gamma X_x)_t = \kappa T_{xx}, \end{cases}$$

which is one of thermo–damping equations corresponding to (*) in one–dimensional case.

As a class of solutions, let me consider the following space: for $t_0 > 0$ I put

$$Z(t_0) = \{(X(t,x), T(x,t)) \mid$$

$$X \in \bigcap_{j=0}^{3} C^j([0,t_0); H^{3-j}), \tag{1.5}$$

$$T \in C^2([0,t_0); L^2) \cap \bigcap_{j=0}^{1} C^j([0,t_0); H^{3-j}), \ \ \partial_t^2 T \in L^2((0,t_0); H^1), \tag{1.6}$$

$$(X_x(t,x), T(t,x)) \in G(\delta) \text{ and } T(t,x) > 0 \text{ for any } (t,x) \in [0,t_0) \times \bar{\Omega}\}. \tag{1.7}$$

I will look for solutions $(X,T) \in Z(\infty)$. Here and hereafter, L^2 denotes the set of all square integrable functions on Ω. H^k denotes the set of all L^2 functions whose distributional derivatives of order up to k also belong to L^2. $C^k(I;B)$ denotes the set of all B–valued k–times continuously differentiable functions on I. $L^2(I;B)$ denotes the set of all B-valued square integrable functions on I.

Now, let me explain the conditions on initial data X_0, X_1, T_0 and a right member f. To do this, for a moment I assume that solutions $(X,T) \in Z(t_0)$ exist. Put

$$X_j(x) = \partial_t^j X(0,x) \ \ \textit{and} \ \ T_j(x) = \partial_t^j T(0,x)$$

which are successively determined through the equations (1.1) and (1.4) in terms of X_0, X_1, T_0, f and their derivatives. For example,

$$X_2(x) = S(X_0'(x), T_0(x)) + f(x),$$

$$T_1(x) = (T_0(x)\frac{\partial \eta}{\partial T}(X_0'(x), T_0(x))^{-1}\{\kappa T_0''(x) - T_0(x)\frac{\partial \eta}{\partial F}(X_0'(x), T_0(x))X_1'(x)\},$$

and so on. According to (1.5) and (1.6), I assume that
(A.5)
$$X_j(x) \in H^{3-j},\ 0 \le j \le 3; \quad T_j(x) \in H^{3-j},\ j = 0,1; \quad T_2(x) \in L^2; \quad f(x) \in H^1.$$

Since

$$T_x(t,x),\ S(X_x(t,x), T(t,x)) \in \bigcap_{j=0}^{1} C^j([0,t_0); H^{2-j}),$$

in view of the trace theorem to the boundary, the boundary condition (N.N) requires the following conditions:

$$\text{(A.6)} \qquad \begin{aligned} &S(X_0'(x), T_0(x)) = 0, \\ &\frac{\partial S}{\partial F}(X_0'(x), T_0(x))X_1'(x) + \frac{\partial S}{\partial T}(X_0'(x), T_0(x))T_1(x) = 0, \\ &T_0'(x) = T_1'(x) = 0 \end{aligned}$$

for $x \in \Omega$. In fact, these conditions come from the facts that $S = S_t = T_x = T_{xt} = 0$ on $\partial\Omega$. (A.6) is called the compatibility condition. In addition, I assume that

$$\text{(A.7)} \qquad \int_0^1 f(x)dx = \int_0^1 X_0(x)dx = \int_0^1 X_1(x)dx = 0.$$

But, (A.7) does not give us any restrictions. In fact, let me consider the compensating function $r(t)$ defined by the formula:

$$r(t) = \int_0^1 X_0(x)dx + t\int_0^1 X_1(x)dx + \frac{t^2}{2}\int_0^1 f(x)dx.$$

Put $\tilde{X}(t,x) = X(t,x) - r(t)$. Then,

$$\begin{aligned} &\int_0^1 \tilde{X}(0,x)dx = \int_0^1 \tilde{X}_t(0,x)dx = 0, \\ &\tilde{X}_{tt} - S(\tilde{X}_x, T)_x = X_{tt} - S(X_x, T)_x - r''(t) = f(x) - \int_0^1 f(x)dx, \\ &\int_0^1 \left(f(x) - \int_0^1 f(x)dx\right) dx = 0. \end{aligned}$$

From these observations, you see that (A.7) actually does not give us any restrictions on initial data and right members.

To find the possible asymptotic behaviour as $t \to \infty$, for unknown functions X_∞ and T_∞ let me consider the stationary problem corresponding to (1.1), (1.4) and (N.N):

$$\begin{aligned} &S(X_\infty'(x), T_\infty(x))' = -f(x) \text{ and } T_\infty''(x) = 0 \quad &&\text{in } \Omega, \\ &S(X_\infty'(x), T_\infty(x)) = T_\infty'(x) = 0 &&\text{on } \partial\Omega. \end{aligned}$$

It follows from the boundary condition that T_∞ is a constant and that

$$S(X'_\infty(x), T_\infty) = -F(x), \tag{1.8}$$

where

$$F(x) = \int_0^x f(y)dy.$$

Note that it follows from (A.7) that $F(0) = F(1) = 0$. A pair $(X_\infty(x), T_\infty)$ satisfying (1.8) is not unique. Another requirement on $(X_\infty(x), T_\infty)$ comes from the following energy conservation law. Integrating (1.2) over $(0,t) \times \Omega$ and noting the relation:

$$X_t f = (X_t F)_x - \frac{\partial}{\partial t}(X_x F),$$

you see that

$$\int_0^1 \left\{ e(X_x(t,x), T(t,x)) + \frac{1}{2} X_t(t,x)^2 + X_x(t,x)F(x) \right\} dx = e_0, \tag{1.9}$$

where

$$e_0 = \int_0^1 \left\{ e(X'_0(x), T_0(x)) + \frac{1}{2} X_1(x)^2 + X'_0(x)F(x) \right\} dx.$$

Since the motion is expected to stop at $t = \infty$, that is $X_t \to 0$ as $t \to \infty$, and since a pair (X_x, T) is expected to converge to (X'_∞, T_∞) as t tends to infinity, (1.9) implies that

$$\int_0^1 \{ e(X'_\infty(x), T_\infty) + X'_\infty(x)F(x) \} dx = e_0. \tag{1.10}$$

Put

$$M_g(F,T) = \frac{\partial S}{\partial F}(F,T)\frac{\partial g}{\partial T}(F,T) - \frac{\partial S}{\partial T}(F,T)\frac{\partial g}{\partial F}(F,T) \text{ for } g = e \text{ and } \eta.$$

Assume that

$$S(1, \tau_0) = 0, \tag{A.8}$$

which means that $(1, \tau_0)$ is an equilibrium state with $f = g = 0$. And then, (A.3), (A.4) and (A.8) imply that

$$M_e(1,\tau_0) = M_\eta(1,\tau_0) = \tau_0 \left\{ -\frac{\partial^2 \psi}{\partial F^2}(1,\tau_0)\frac{\partial^2 \psi}{\partial T^2}(1,\tau_0) + \frac{\partial^2 \psi}{\partial F}\partial T(1,\tau_0)^2 \right\} > 0. \tag{1.11}$$

In view of (1.11), by the implicit function theorem you can prove the following lemma concerning the unique existence of a pair $(X_\infty(x), T_\infty)$ satisfying (1.8) and (1.10).

LEMMA 1.2. *Suppose that* (A.3), (A.4), (A.5), (A.7) *and* (A.8) *hold. Then, for any* $\sigma > 0$ *there exists a* $\kappa > 0$ *such that if*

$$||(X_0', T_0) - (1, \tau_0)||_\infty + ||X_1|| + ||f||_1 < \kappa$$

then there exist a $X_\infty(x) \in H^3$ *and a constant* $T_\infty > 0$ *satisfying* (1.8) *and* (1.10) *and the following conditions:*

$$||X_\infty' - 1||_2 + |T_\infty - \tau_0| < \sigma \ \text{ and } \ (X_\infty'(x), T_\infty) \in G(\delta/2) \quad \text{for all } x \in \bar{\Omega}.$$

In particular, $S(X_\infty'(x), T_\infty) = 0$ *for* $x \in \partial\Omega$.

Here and hereafter, $|| \cdot ||$ denotes the usual L^2–norm on Ω and put

$$||v||_k = \left\{ \sum_{j=0}^{k} ||\frac{d^j v}{dx^j}||^2 \right\}^{1/2} \quad \text{and} \quad ||v||_\infty = \sup_{x \in \Omega} |v(x)|.$$

To state a main result exactly, let me introduce an additional notation. Put

$$\begin{aligned}
&u(t,x) = X(t,x) - X_\infty(x), \ \ \theta(t,x) = T(t,x) - T_\infty, \\
&N(t) = \sup_{0<s<t} ||\bar{D}^2(u_x, u_t, \theta)(s,\cdot)||, \\
&N_\alpha(t) = \sup_{0<s<t} e^{\alpha s} \left\{ ||\bar{D}^2(u_x, u_t, \theta)(s,\cdot)|| + ||(\theta_{xxt}, \theta_{xxx})(s,\cdot)|| \right\}, \\
&M_\alpha(t) = \left\{ \int_0^t e^{2\alpha s} ||(D^2 u, D^3 u, D^1\theta, D^2\theta, \theta_{xtt}, \theta_{xxt}, \theta_{xxx})(s,\cdot)||^2 ds \right\}^{1/2}, \\
&E_0 = ||X_0' - X_\infty'||_2 + ||T_0 - T_\infty||_3 + \sum_{j=1}^{3} ||X_j||_{3-j} + ||T_1||_2 + ||T_2||.
\end{aligned}$$

Here and hereafter, I use the following symbols:

$$D^k u = (\frac{\partial^k u}{\partial t^j \partial x^{j-k}}, j = 0, 1, \cdots, k) \ \text{ and } \ \bar{D}^k u = (u, D^1 u, \ldots, D^k u).$$

Under these preparations, I can state the main result of this section which was proved by Kawashima and Shibata [3] in the following way.

THEOREM 1.3. *Suppose that* (A.1)–(A.8) *hold and that* $T_0(x) > 0$ *for* $x \in \bar{\Omega}$. *Then, there exists an* $\epsilon > 0$ *such that if* $||(X_0', T_0) - (1, \tau_0)||_\infty + E_0 + ||f||_1 < \epsilon$, *then the problem* (1.1), (1.2), (N.N) *and* (1.3) *admits a unique solution* $(X(t,x), T(t,x)) \in Z(\infty)$ *satisfying the estimate:*

$$N_\alpha(t)^2 + M_\alpha(t)^2 \leq C E_0^2 \tag{1.12}$$

for suitable positive constants α and C.

A Sketch of A Proof of Theorem 1.3.

The first step of a proof of Theorem 1.3 is a local in time existence theorem. I quote the result due to W. Dan [1], which treated the Neumann problem for more general quasilinear hyperbolic–parabolic coupled systems in any dimensional space.

Local in time existence theorem. *Suppose that* (A.2), (A.3), (A.4), (A.5) *and* (A.6) *hold. In addition, suppose that*

$$(X_0'(x), T_0(x)) \in G(\delta/2) \text{ and } T_0(x) > 0 \qquad \text{for } x \in \bar{\Omega},$$

$$f,\ g \in \bigcap_{j=0}^{1} C^j([0,t_0); H^{1-j}) \text{ and } \partial_t^2 f,\ \partial_t^2 g \in L^2((0,t_0); L^2).$$

Let B be a positive number such that

$$\sum_{j=0}^{3} \|X_j\|_{3-j} + \sum_{j=0}^{1} \|T_j\|_{3-j} + \|T_2\| + \sum_{j=0}^{1} \sup_{0\le s\le t_0} \|(\partial_t^j f, \partial_t^j g)(s,\cdot)\|_{1-j} + \left\{ \int_0^{t_0} \|(\partial_t^2 f, \partial_t^2 g)(s,\cdot)\|^2 ds \right\}^{1/2} \le B.$$

Then, there exists a time $t_1 \in (0, t_0)$ depending only on B essentially such that the problem (1.1), (1.2), (N.N) *and* (1.3) *admits a unique solution $(X,T) \in Z^3(t_1)$ satisfying the condition:*

$$(X_x(t,x), T(t,x)) \in G(2\delta/3) \qquad \text{for all } (t,x) \in [0,t_1] \times \bar{\Omega}.$$

Combining this local in time existence theorem and *a priori* estimates of local in time solutions, we can extend local in time solutions to any time interval, and then Theorem 1.3 is established. Therefore, the main step of the proof in Kawashima and Shibata [3] is to show the following *a priori* estimates of a local in time solution $(X,T) \in Z^3(t_0)$.

A priori estimates. *There exist positive constants C, α and σ such that* (1.12) *holds provided that*

$$N(t) \le \sigma \qquad \text{for } 0 \le t \le t_0, \tag{1.13}$$

$$\|(X_\infty', T_\infty) - (1, \tau_0)\|_\infty + \|X_\infty''\|_\infty + \|X_\infty''\|_1 + \|f\| \le \sigma. \tag{1.14}$$

Our derivation of (1.12) is divided into eight steps.

STEP 1. *I verify the relations:*

$$\|u_t(t,\cdot)\| \le C\|u_{tx}(t,\cdot)\|, \tag{1.15}$$

$$\|(u_x,\theta)(t,\cdot)\| \le C\|(u_t,u_x,\theta)_x(t,\cdot)\|, \tag{1.16}$$

provided that σ is small enough. Here and hereafter, the letter C denotes various constants independent of α and σ.

To prove Step 1, we use the following Poincaré's inequalities:

$$\|v\| \le C\left\{\left|\int_0^1 p(x)v(x)dx\right| + \|v'\|\right\} \tag{1.17}$$

for $v \in H^1$, where $p(x) \in L^2$ such that $\int_0^1 p(x)dx \ne 0$;

$$\|v\| \le C\{< v > + \|v'\|\} \tag{1.18}$$

for $v \in H^1$, where $< v >= |v(0)| + |v(1)|$. (A.7) implies that $\int_0^1 u_t(t,x)dx = 0$, which combined with (1.17) implies (1.15). Put $S_\infty = S(X'_\infty(x), T_\infty)$. Since $S - S_\infty = 0$ on $\partial\Omega$ as follows from (N.N), (1.8) and the fact: $F(0) = F(1) = 0$, applying (1.18) to $S - S_\infty$ implies that

$$\|(S_F^0 u_x + S_T^0\theta)(t,\cdot)\| \le C\{\|(u_x,\theta)_x(t,\cdot)\| + \sigma\|(u_x,\theta)(t,\cdot)\|\}, \tag{1.19}$$

where we have used the formula: $S - S_\infty = S_F^0 u_x + S_T^0\theta$ which is obtained by the Taylor expansion, that is

$$S_G^0 = \int_0^1 \frac{\partial S}{\partial G}((X'_\infty(x), T_\infty) + \ell(u_x(t,x), \theta(t,x)))d\ell \text{ for } G = F \text{ and } T.$$

In the same manner, we can write: $e(X_x, T) = e(X'_\infty(x), T_\infty) + e_F^0 u_x + e_T^0\theta$. Inserting this formula into (1.9) and using (1.10), we see that

$$\int_0^1 \{e_F^0 u_x + e_T^0\theta + \frac{1}{2}u_t^2 + u_x F\}dx = 0, \tag{1.20}$$

which combined with (1.17) and (1.15) implies that

$$\|(e_F^0 u_x + e_T^0\theta)(t,\cdot)\| \le C\{\sigma\|(u_x,\theta)(t,\cdot)\| + \|(u_x,u_t,\theta)_x(t,\cdot)\|\}. \tag{1.21}$$

Let us define the matrix U by:

$$U = \begin{pmatrix} S_F^0 & e_F^0 \\ S_T^0 & e_T^0 \end{pmatrix}.$$

(1.11) guarantees that the determinant of U is positive provided that σ is small enough, so that combining (1.19) and (1.21) implies (1.16).

Note that (1.1) can be rewritten in the form:

$$u_{tt} - (S_F^0 u_x + S_T^0 \theta)_x = 0 \tag{1.22}$$

with $S_F^0 u_x + S_T^0 \theta = 0$ on $\partial\Omega$. Differentiating (1.22) and (1.4) k–times ($k = 0$, 1 and 2) with respect to t and multiplying the resulting equations by $\partial_t^{k+1} u$ and $(T_\infty + \theta)^{-1} \partial_t^k \theta$, we can derive the following step.

STEP 2. *We have*

$$e^{2\alpha t} \|\bar{\partial}_t^2 (u_t, u_x, \theta)(t, \cdot)\|^2 + c_1 \int_0^1 e^{2\alpha s} \|\bar{\partial}_t^2 \theta_x (s, \cdot)\|^2 ds \leq C R_\alpha(t) \tag{1.23}$$

for a suitable constant $c_1 > 0$, where

$$R_\alpha(t) = E_0^2 + (\sigma + \alpha) M_\alpha(t)^2 + \sigma N_\alpha(t)^2 \ \text{ and } \ \bar{\partial}_t^k v = (v, \partial_t v, \dots, \partial_t^k v).$$

In the derivation of (1.12), we use the identity:

$$e^{2\alpha t} g(t) = \int_0^t e^{2\alpha s} \frac{dg}{ds}(s) ds + g(0) + 2\alpha \int_0^t e^{2\alpha s} g(s) ds.$$

STEP 3. *For σ small enough, I verify the relation:*

$$N_\alpha(t)^2 \leq C R_\alpha(t). \tag{1.24}$$

In view of (1.23), we have to estimate the terms: θ_x, θ_{xt}, u_{xx}, θ_{xx}, u_{xxx}, u_{xxt}, θ_{xxx} and θ_{xxt}. From (1.4) it follows that

$$(Q \partial_t^k \theta_x)_x = -k (Q_t \theta_x)_x + \partial_t^k ((T_\infty + \theta) \eta_t) \ \text{ for } k = 0 \text{ and } 1.$$

Multiplying this by $\partial_t^k \theta$ and integration by parts and using the fact that $\|Q_t\|_\infty \leq C\sigma$, which follows from Sobolev's inequality and (1.13), we have

$$|(Q\theta_x, \theta_x)| + |(Q\theta_{xt}, \theta_{xt})| \leq C\{\sigma \|(\theta_x, \theta_{xt})(t, \cdot)\|^2 + \|\bar{\partial}_t^2 (u_t, u_x, \theta)(t, \cdot)\|^2\},$$

where $(\cdot, \cdot)$ denotes the usual L^2–innerproduct, which combined with (1.23) implies that

$$e^{2\alpha t} \|(\theta_x, \theta_{xt})(t, \cdot)\|^2 \leq C R_\alpha(t). \tag{1.25}$$

From (1.4) and (1.22) it follows that

$$S_F^0 u_{xx} = u_{tt} - \{S_T^0 \theta_x + (S_F^0)_x u_x + (S_T^0)_x \theta\}, \tag{1.26}$$

$$Q\theta_{xx} = -Q_x \theta_x + (T_\infty + \theta)(\eta_T \theta_t + \eta_F u_{xt}). \tag{1.27}$$

By using these formulae, (1.23) and (1.25) we have

$$e^{2\alpha t}\|(u_{xx},\theta_{xx})(t,\cdot)\| \le CR_\alpha(t). \tag{1.28}$$

Differentiating (1.26) and (1.27) with respect to t and x and using (1.23), (1.25) and (1.28), we have

$$e^{2\alpha t}\|(u_{xxx},u_{xxt},\theta_{xxx},\theta_{xxt})(t,\cdot)\|^2 \le CR_\alpha(t). \tag{1.29}$$

Combining (1.23), (1.25), (1.28) and (1.29) implies (1.24).

Now, I am going to estimate $M_\alpha(t)$. The estimation for $\|\bar{\partial}_t^2\theta_x(t,\cdot)\|^2$ was already obtained.

STEP 4. *I verify the relation:*

$$\int_0^t e^{2\alpha s}\|\theta_{xxx}(s,\cdot)\|^2 ds \le C\left\{\int_0^t e^{2\alpha s}\|u_{xxt}(s,\cdot)\|^2 ds + R_\alpha(t)\right\}; \tag{1.30}$$

$$\int_0^t e^{2\alpha s}\|(\theta_{xx},\theta_{xxt})(s,\cdot)\|^2 ds \le \delta\int_0^t e^{2\alpha s}\|u_{xxt}(s,\cdot)\|^2 ds + C\delta^{-1}R_\alpha(t) \tag{1.31}$$

for $\delta\in(0,1)$.

Differentiating (1.27) once with respect to x, we have (1.30) immediately. Using (1.27) and the formula of differentiation of (1.27) once with respect to t, we have also (1.31) immediately.

STEP 5. *I verify the relation:*

$$\int_0^t e^{2\alpha s}\|D^3u(s,\cdot)\| ds \le C\left\{\int_0^t e^{2\alpha s}\|u_{xxt}(s,\cdot)\|^2 ds + R_\alpha(t)\right\}. \tag{1.32}$$

Put $K_L = \partial K/\partial L$ for $K = S$ and η and $L = F$ and T. First, I shall use the following representation formula for u_{xtt}:

$$Vu_{xtt} = S_{tt} + (((T_\infty+\theta)\eta_T)^{-1}S_T(Q\theta_x)_x)_t - V_t u_{xt}, \tag{1.33}$$

where

$$V = \eta^{-1}M_\eta(X'_\infty + u_x, T_\infty+\theta) \ \text{(cf. (1.11))}.$$

This representation follows from the differentiation with respect to t of the formula:

$$S_t = Vu_{xt} + ((T_\infty+\theta)\eta_T)^{-1}S_T(Q\theta_x)_x, \tag{1.34}$$

which follows from the combination of the following two formulae:

$$S_t = S_F u_{xt} + S_T \theta_t \text{ and } \theta_t = ((T_\infty + \theta)\eta_T)^{-1} \{(Q\theta_x)_x - (T_\infty + \theta)\eta_F u_{xt}\}.$$

Since $V > 0$ for σ small enough, multiplying (1.33) by u_{xtt} and noting the identity:

$$(S_{tt}, u_{xtt}) = \frac{d}{dt}(S_t, u_{xtt}) + (S_{xt}, u_{ttt}) = \frac{d}{dt}(S_t, u_{xtt}) + (S_{xt}, S_{xt}),$$

we have

$$\int_0^t e^{2\alpha s} ||u_{xtt}(s,\cdot)||^2 ds \le C\left\{\int_0^t e^{2\alpha s} ||u_{xxt}(s,\cdot)||^2 ds + R_\alpha(t)\right\}.$$

Differentiating (1.26) once with respect to x and using the formula: $u_{ttt} = S_{xt}$, we have easily that

$$\int_0^t e^{2\alpha s} ||(u_{xxx}, u_{ttt})(s,\cdot)||^2 ds \le C\left\{\int_0^t e^{2\alpha s} ||u_{xxt}(s,\cdot)||^2 ds + R_\alpha(t)\right\}, \tag{1.35}$$

where we have used the relation: $||u_{tt}|| \le C||u_{xtt}||$, which follows from the fact that $\int_0^1 u_{tt}(t,x)dx = 0$ and (1.17).

One of the main points of the proof is the following two steps and the idea goes back to the paper due to Muños Rivera [4].

STEP 6. *I verify the relation:*

$$\begin{aligned} \int_0^t e^{2\alpha s} ||(D^3 u, \theta_{xx}, \theta_{xxx}, \theta_{xxt})(s,\cdot)||^2 ds \\ \le \mu \int_0^t e^{2\alpha s} < u_{xxt}(s,\cdot) >^2 ds + C\mu^{-4} R_\alpha(t) \end{aligned} \tag{1.36}$$

for $\mu > 0$ small enough.

In view of (1.30), (1.31) and (1.32), we shall estimate $||u_{xxt}(s,\cdot)||$. To do this, we use the formula:

$$Q\theta_{xxx} - (T_\infty + \theta)\eta_T \theta_{xt} - (T_\infty + \theta)\eta_F u_{xxt} + g_1 = 0, \tag{1.37}$$

where

$$g_1 = Q_x \theta_{xx} + Q_{xx}\theta_x - ((T_\infty + \theta)\eta_T)_x \theta_t - ((T_\infty + \theta)\eta_F)_x u_{xt},$$

which follows from the differentiation of (1.27) with respect to x. Since

$$\begin{aligned} (Q\theta_{xxx}, u_{xxt}) =& < Q\theta_{xx}, u_{xxt} > -(Q_x\theta_{xx}, u_{xxt}) \\ & - \frac{d}{dt}(Q\theta_{xx}, u_{xxx}) + (Q_t\theta_{xx}, u_{xxx}) + (Q\theta_{xxt}, u_{xxx}) \end{aligned}$$

where $< u, v >= u(1)v(1) - u(0)v(0)$, multiplying (1.37) by u_{xxt} implies that

$$\text{(1.38)} \qquad \int_0^t e^{2\alpha s}\|u_{xxt}(s,\cdot)\|^2 ds \le \int_0^t e^{2\alpha s}| < Q\theta_{xx}, u_{xxt} > |ds \\ + \int_0^t e^{2\alpha s}|(Q\theta_{xxt}, u_{xxx})|ds + CR_\alpha(t).$$

Choosing $\delta > 0$ small enough in (1.31) and using Schwarz's inequality and (1.32), the second term of the right-hand side of (1.38) can be absorbed by the left-hand side of (1.38). Since

$$< \theta_{xx}(t,\cdot) >^2 \le \|\theta_{xx}(t,\cdot)^2\|_\infty \le C\int_0^1 |\bar{\partial}_x^1(\theta_{xx}(t,\cdot)^2)|dx \\ \le C\{\|\theta_{xx}(t,\cdot)\|^2 + \|\theta_{xx}(t,\cdot)\|\|\theta_{xxx}(t,\cdot)\|\}$$

where $\bar{\partial}_x^k v = (v, \partial_x v, \dots, \partial_x^k v)$ and we have used Sobolev's inequality:

$$\|v\|_\infty \le C\int_0^1 |\bar{\partial}_x^1 v(x)|dx,$$

we have

$$\int_0^t e^{2\alpha s}| < Q\theta_{xx}, u_{xxt} > |ds \\ \le \mu\int_0^t e^{2\alpha s} < u_{xxt}(s,\cdot) >^2 ds + C\mu^{-1}\int_0^t e^{2\alpha s} < \theta_{xx}(s,\cdot) >^2 ds \\ \le \mu\int_0^t e^{2\alpha s} < u_{xxt}(s,\cdot) >^2 ds + C\mu^{-1}(1+\delta^{-1/2})\int_0^t e^{2\alpha s}\|\theta_{xx}(s,\cdot)\|^2 ds \\ + C\mu^{-1}\delta^{1/2}\int_0^t e^{2\alpha s}\|\theta_{xxx}(s,\cdot)\|^2 ds.$$

Inserting this inequality into (1.38) and using (1.30) and (1.31), we have (1.36) by choosing μ and δ in such a way that $\mu = \kappa\delta^{1/2}$ for a suitably small constant $\kappa > 0$.

STEP 7. *I verify the relation:*

$$\text{(1.39)} \qquad \int_0^t e^{2\alpha s} < u_{xxt}(s,\cdot) >^2 ds \\ \le C\left\{\int_0^t e^{2\alpha s}\|(D^3u, \theta_{xx}, \theta_{xxx}, \theta_{xxt})(s,\cdot)\|^2 ds + R_\alpha(t)\right\}.$$

Differentiating (1.1) with respect to t and x implies that

$$\text{(1.40)} \qquad u_{xxxt} = S_F^{-1}\{u_{xttt} - S_T\theta_{xxt} - g_2\}$$

where

$$g_2 = 2\left((S_F)_x u_{xxt} + (S_T)_x \theta_{xt}\right) + (S_F)_{xx} u_{xt} + (S_T)_{xx}\theta_t.$$

Put $q(x) = x - \frac{1}{2}$ and then we have

$$\begin{aligned}(u_{xxxt}, qu_{xxt}) &= \frac{1}{4} < u_{xxt} >^2 - \frac{1}{2}\|u_{xxt}\|^2;\\ (S_F^{-1}u_{xttt}, qu_{xxt}) &= \frac{d}{dt}(S_F^{-1}u_{xtt}, qu_{xxt}) - ((S_F^{-1})_t u_{xtt}, qu_{xxt})\\ &\quad + \frac{1}{2}\left((S_F^{-1}q)_x u_{xtt}, u_{xtt}\right) - \frac{1}{4}\left\{(S_F^{-1}u_{xtt}^2)(t,1) + (S_F^{-1}u_{xtt}^2)(t,0)\right\}.\end{aligned}$$

Therefore, multiplying (1.40) by qu_{xxt} implies that

$$\begin{aligned}(1.41)\quad &\frac{1}{4}\left\{\left((1+S_F^{-1})u_{xtt}^2\right)(t,1) + \left((1+S_F^{-1})u_{xtt}^2\right)(t,0)\right\}\\ &\qquad \le C\left\{\|\theta_{xxt}\|^2 + \|D^3u\|^2 + \|g_2\|^2\right\} + \frac{d}{dt}(S_F^{-1}u_{xtt}, qu_{xxt}).\end{aligned}$$

Multiplying (1.41) by $e^{2\alpha t}$, integrating the resulting equation over $(0,t)$ and using (1.24), we have (1.39).

Combining (1.36) and (1.39) and choosing μ small enough, we arrive at the relation:

$$(1.42)\qquad \int_0^t e^{2\alpha s}\|(D^3u, \theta_{xx}, \theta_{xxx}, \theta_{xxt})(s,\cdot)\|^2 ds \le CR_\alpha(t).$$

STEP 8. *I verify the relation:*

$$(1.43)\qquad \int_0^t e^{2\alpha s}\|(D^2u, \theta_t, \theta_{tt})(s,\cdot)\|^2 ds \le CR_\alpha(t).$$

We already knew the estimate: $\|u_{tt}\| \le C\|u_{ttx}\|$. The estimate of $\|u_{xx}\|$ follows from the formula: $u_{xx} = (S_F^0)^{-1}\{u_{tt} - S_T^0\theta_x - (S_F^0)_x u_x - (S_T^0)_x\theta\}$. Since $S_t = 0$ on $\partial\Omega$ as follows from $S = 0$ on $\partial\Omega$, by (1.18) we have

$$\|S_t\|^2 \le C\|S_{xt}\|^2 \le C\{\|(u_{xxt}, \theta_{xt})(t,\cdot)\|^2 + \sigma\|(u_{xt}, \theta_t)(t,\cdot)\|^2\},$$

which combined with (1.34) implies that the estimate of $\|u_{xt}\|$. Finally, the estimates of $\|\theta_t\|$ and $\|\theta_{tt}\|$ follow from the formula:

$$\theta_t = ((T_\infty + \theta)\eta_T)^{-1}\{(Q\theta_x)_x - (T_\infty + \theta)\eta_F u_{xt}\}$$

and the formula of its differentiation, respectively.

Summing up, we have obtained

$$N_\alpha(t)^2 + M_\alpha(t)^2 \le C\{E_0^2 + (\sigma + \alpha)M_\alpha(t)^2 + \sigma N_\alpha(t)^2\},$$

and then choosing σ and α small enough, we have (1.12).

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